PRACTICE Exam 2

Name: ______________________________

NOTES:

- This is a practice exam. It will not be graded. The aim is to give you a sense of the types of questions that may appear on the exam.
- As a reminder, this is not an exhaustive list of types of problems, anything through probability (class on April 17) is fair game.
- I recommend preparing your notes sheet before attempting these problems, then revising it to see if it needs to be adapted.
- Attempt all problems. On the exam, it is possible to receive partial credit for showing your work.

Reference Page

This page does not have any questions. Instead it is a reference for relevant definitions and theorems for this exam.

A *graph* G is a set of vertices and a set of edges. We usually write $G = (V, E)$.

A *simple graph* has no loops or parallel edges.

The *degree* of vertex u is the number of edge-ends at u.

The *handshaking lemma* says that for a graph $G = (V, E)$

$$2|E| = \sum_{x \in V} d(x) \ .$$

A *tree* is a graph that is connected and has no polygons.

A graph G is bipartite if the set of vertices may be partitioned into two independent sets R and B such that each vertex is in exactly one of R and B and every edge in G joins an edge from R to B.

A graph G is *planar* if it can be drawn on a sheet of paper where no two edges cross.

Euler's Formula says that if G is a connected planar graph, then in any drawing of G, $|V| - |E| + |F| = 2$.

A *relation* on a set S is any set of ordered pairs of elements of S.

A relation R on S is *reflexive* if aRa for all $a \in S$.

A relation R on S is called *transitive* if whenever aRb and bRc, then aRc, for all $a, b, c \in S$.

A relation R on S is called *symmetric* if whenever aRb then bRa, for all $a, b \in S$.

A relation R on S is called *antisymmetric* if whenever aRb and bRa it means $a = b$, for all $a, b \in S$.

A relation R on S has the *comparability property* if either aRb or bRa or $a = b$ for all $a, b \in S$.

A relation R is an *equivalence relation* if it is reflexive, symmetric, and transitive.

A relation R is an *order relation* if it is antisymmetric and transitive. It's a *partial order* if it's also reflexive. A partial order is a *total order* if it also has the comparability property.

The *expected value* of a random variable X is $E[X] = \sum_{v \in V} v \cdot Pr[X = v]$, where V is the set of values which X takes on.

The *conditional probability* of B given A is only defined when $Pr[A] > 0$ and is given by

$$Pr[B|A] = \frac{Pr[A \text{ and } B]}{Pr[A]}$$

Events A and B are *independent* if $Pr[A \text{ and } B] = Pr[A] \cdot Pr[B]$.

Events A and B are *mutually exclusive* if $Pr[A \text{ and } B] = 0$.

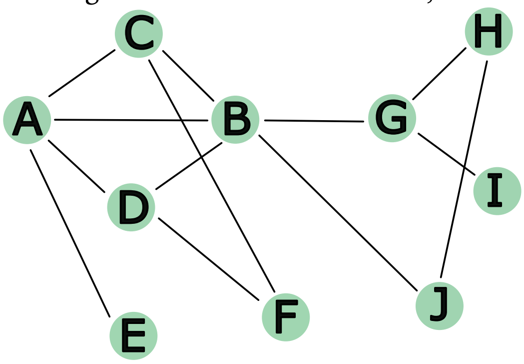
Problem 1	Score: /

Prove the following claim.

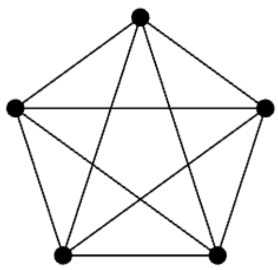
Claim 1. *Suppose T is a tree where there is some vertex u with $deg(u) = k > 1$. Then T must have at least k vertices with degree 1.*

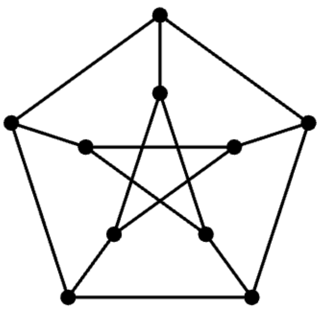
Problem 2	Score: /

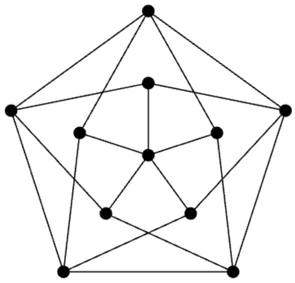
a) Compute spanning trees by each of depth-first and breadth-first traversals for the following graph starting at vertex A. In both cases, demonstrate the traversal progress with a table.

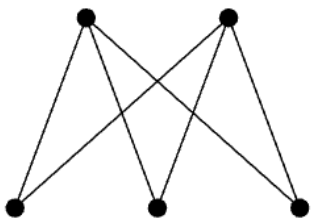


b) Which of the following graphs are planar?









Problem 3 Score: /

Consider a set $S = \mathcal{P}(\{2, 3, 4, 5\})$, which is defined as a powerset. (Refer back to our chapter on set notation if you forget what this is.) For each of the following relations, determine whether they are partial orders, total orders, or neither. Justify your answers.

a) $(A, B) \in R_1 \Leftrightarrow \sum_{a \in A} a \leq \sum_{b \in B} b$

b) $(A, B) \in R_2 \Leftrightarrow A \subseteq B$

Problem 4	Score: /

For a relation R on a set S, define the inverse of R, R^{-1} to be the set of every element of R in reverse order. That is, $R^{-1} = \{(b, a) \mid (a, b) \in R\}$. For each of the following claims, either prove it is true or disprove it by giving a counterexample.

a) R is antisymmetric if and only if $R \cap R^{-1} \subseteq \{(a, a) \mid a \in S\}$.

b) R is reflexive if and only if R^{-1} is reflexive.

Problem 5	Score: /

Suppose that we have two coins. One is a regularly weighted fair coin meaning that when we flip it, each side has equal probability of coming up. The second coin is a biased coin, where heads has a probability of p, and tails has a probability of $1 - p$.
Consider the following experiment. We put both of the coins in a bag. We pull one of the coins out of the bag (uniformly at random) and flip it.

a) If $p = 2/3$, what is $Pr[\text{the pulled coin is biased} \mid \text{we observe a single heads}]$?

b) If $p = 3/4$, what is $Pr[\text{the pulled coin is biased} \mid \text{we observe HHHT}]$?

c) Say $p = 3/4$ and we repeatedly conduct this experiment until a tails comes up. What is the expected value of this process?

Problem 6	Score: /

Say we have a hash table with m slots, which currently has $i-1$ slots filled (with $0 < i < m$). Let X_i be the random variable which denotes the number of elements that are hashed until one more cell is filled. What is $E[X_i]$?