PRACTICE Exam 1

Name: _____

NOTES:

- This is a practice exam. It will not be graded. The aim is to give you a sense of the types of questions that may appear on the exam.
- As a reminder, this is not an exhaustive list of types of problems, anything through graphs (class on February 20) is fair game.
- I recommend preparing your notes sheet before attempting these problems, then revising it to see if it needs to be adapted.
- Attempt all problems. On the exam, it is possible to receive partial credit for showing your work.

Reference Page

This page does not have any questions. Instead it is a reference for relevant definitions and theorems for this exam.

A *graph* G is a set of vertices and a set of edges. We usually write $G = (V, E)$.

A *simple graph* has no loops or parallel edges.

The *degree* of vertex u is the number of edge-ends at u.

A *path* is a sequence of the form

$$\pi = (v_0, e_1, v_1, e_2, \ldots, e_k, v_k)$$

where each e_i is an edge joining vertex v_{i-1} to vertex v_i.

A *Euler path* of a graph G is a path where each edge of G occurs exactly once.

A *Hamiltonian path* of a graph G is a path where each vertex occurs exactly once.

The *handshaking lemma* says that for a graph $G = (V, E)$

$$2|E| = \sum_{x \in V} d(x) \ .$$

Problem 1	Score: /

Let S be a set with $|S| = n$ and T be a set with $|T| = m$.

a) What is the smallest and largest possible cardinalities of $S \cup T$?

b) What is the smallest and largest possible cardinalities of $S \cap T$?

c) What is the smallest and largest possible cardinalities of $S \backslash T$?

Problem 2	Score: /

You work for a company who requires its employees to have a special short password, sort of like a PIN, for certain systems. This password should contain only letters, either lowercase or uppercase, and should be exactly length four.
Note: You may leave your answers un-simplified, just be sure to justify how you arrived at your answers.

a) How many possible passwords are there?

b) How many possible passwords are there that only use lowercase letters?

c) How many possible passwords are there that use exactly 2 uppercase letters, and no repetitions of letters across lower or upper case (aka if T is included, another T may not be included and t may not be included).

d) How many possible passwords are there that use at least one uppercase letter and at least one lowercase letter?

Problem 3	Score: /

Logic.

a) Use a truth table to determine the validity of the following argument form:

$$\begin{array}{l} \sim P \vee Q \\ \sim [R \wedge (\sim Q)] \\ \hline \therefore (P \wedge R) \rightarrow Q \end{array}$$

b) Use a truth table to determine whether the following two logical statements are equivalent.

$(P \vee Q) \rightarrow R$ and $(P \rightarrow R) \vee (Q \rightarrow R)$

Problem 4	Score: /

Prove the following claim

Claim 1. *Let n be any integer. Then $n^3 - n$ is divisible by 3.*

Problem 5	Score: /

Prove the following for all $n \geq 1$ using induction:

$$1 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}\,.$$

Problem 6	Score: /

You may take the following lemma as fact (you don't need to prove it):

Lemma 1. *An even number plus an even number is an even number. An odd number plus another odd number is an even number. An odd number plus an even number is an odd number.*

Prove the following statement:

Claim 2. *In any graph the number of vertices with odd degree must be even.*